“From Equal Radicals to Fermat’s Last Theorem (with radicals)”

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1.0 AN OBSERVATION

The following project was inspired by a personal finding during Test #1 in Math 510A [1], taught by Prof. Dr. W. Watkins. In particular, problem #2 in that test asked to **"…find a polynomial with integral (integer) coefficients that annihilates**

$$\alpha = \sqrt{5} - \sqrt{2}$$

The solution to the given problem was pretty simple, as follows:

$$\alpha + 2 = \sqrt{5}$$

By squaring both sides:

$$\alpha^2 + 2\sqrt{2}\alpha + 2 = 5$$

or:

$$(2\sqrt{2}\alpha)^2 = (3 - \alpha^2)^2$$

or

$$8\alpha^2 = 9 - 10\alpha^2 - \alpha^4$$

or

$$\alpha^4 - 14\alpha^2 + 9 = 0$$

Therefore, α is a root of the polynomial:

$$P(x) = x^4 - 14x^2 + 9$$

Furthermore, during Test #1, I tried to verify my finding and I have made the observation that if I take:

$$y = x^2$$

then P(x) from above becomes:

$$Q(y) = y^2 - 14y + 9$$

Thus, by the quadratic formula, the roots of Q(y) are:

$$y_{1,2} = ½[14 \pm \sqrt{(14^2 - 4 * 9)}] = ½ (14 \pm \sqrt{160}) = 7 - 2\sqrt{10}$$

Therefore, the roots of the initial equation:

$$P(x) = x^4 - 14x^2 + 9$$

are:

$$x_{1,2,3,4} = \pm\sqrt{(7 - 2\sqrt{10})}$$

And now comparing this finding with the solution to the initial problem, I realized that

$$\alpha = \sqrt{5} - \sqrt{2}$$

must be one of the four previous values for x, even though apparently they don't look like a good match.

1.1 AN HYPOTHESIS

Thus, the hypothesis is that:

$$\sqrt{5} - \sqrt{2} = \pm\sqrt{(7 - 2\sqrt{10})} = 0.821854415\ldots$$

which is simple to verify, by squaring both sides of the identity:

$$5 + 2 - 2\sqrt{10} = (7 - 2\sqrt{10})$$

or

$$7 - 2\sqrt{10} = 7 - 2\sqrt{10}$$

Next, I have made the observation that if I replace '-' by '+' between the radicals the identity still stands, i.e., :

$$\sqrt{5} + \sqrt{2} = \pm\sqrt{(7 + 2\sqrt{10})}$$

which, again, is quite simple to verify, by squaring both sides of the identity.

Furthermore, being inspired by the "interesting" finding, I have said, how about this identity:

$$\boxed{\sqrt{m} \pm \sqrt{n} = \pm\sqrt{[(m + n) \pm 2\sqrt{m*n}]}}$$

where m, n are natural numbers ($m = n = 0$ also verify, as a trivial solution) and which again is simple to verify, by squaring both sides of the identity:

$$m + n \pm 2\sqrt{m*n} = (m + n) \pm 2\sqrt{m*n}$$

Right after submitting my test paper to Prof. Watkins I spoke to him about my finding during the test, more like a curiosity – not even close to claim I have made any discovery. Next class session, Prof. Watkins said that's actually an interesting finding, thus he suggested me to make it a project paper for Math 10A/B courses pair. Furthermore, he suggested the title of the paper 'Equal Radicals'(2) and have added another similar problem that I could further extend.

2.0 ANOTHER RELATED PROBLEM

Here is the second (related) problem (as suggested by Prof. Dr. Watkins[2]):

"Let *a, b,* and *c* be positive integers (*a* = *b* = *c* = 0 also verify, as a trivial solution). What are the conditions on *a, b,* and *c* must hold in order to satisfy the identity:

$$\sqrt{a} = \sqrt{b} + \sqrt{c}\,\text{?"}$$

The solution is fairly simple, as follows:

Let:

$$\alpha = \sqrt{a} \text{ and } \beta = \sqrt{b} + \sqrt{c}$$

Thus we must find *a, b*, and *c*, such that:

$$\alpha = \beta$$

Through few simple computational steps (similar to those performed in paragraph 1.0 above), we find that the polynomial:

$$P(x) = x^2 - a$$

annihilates α, and the polynomial:

$$Q(x) = x^4 - 2(b + c)x^2 + (b - c)^2$$

Annihilates β.

Now, if:

$$\alpha = \beta$$

Then *a* is a root of Q(x), thus:

$$a^2 - 2(b + c)\,a + (b - c)^2 = 0$$

And using the quadratic formula for the above equation with *a* as unknown:

$$\boxed{(*) \qquad a_{1,2} = b + c \pm 2\sqrt{bc}}$$

Since *a, b*, and *c* must be integers, then $\sqrt{bc}$ must also be an integer and hence *bc* must be a perfect square.

At this point Prof. Dr. Watkins[2] said: **"More can be said (about *b* and *c*)."** – thus leaving me with the opportunity to further explore that relationship. Therefore, as part of the same project, I started analyzing what the relationship between *b* and *c* must be such as identity (*) above can be satisfied amongst integers *a, b* and *c*.

2.1 ANOTHER FINDING

I've heard (and saw working in the classroom) from Prof. Watkins that *Mathematica*[11] is a nice package to have these days of computing era, helping tremendously with all kind of complex computations and also eliminating the typical human error in calculations (that I'm guilty of quite often). And, purchasing a copy is high on my agenda. Meanwhile, trying to get this project proposal out, I found a good use for *Microsoft Excel* to look at the product of integers *bc* and see if I can notice any pattern - little bit of cheating… - that would help be to more precise when say what *b* and *c* could be in order to satisfy (*) above.

Therefore, the finding was that *b* and *c* must be of this form:

$$\boxed{\mathbf{b = b_1{}^{p}{}_1 * b_2{}^{p}{}_2 * \ldots * b_n{}^{p}{}_n (* d_1{}^{2q}{}_1 * d_2{}^{2q}{}_2 * \ldots * d_m{}^{2q}{}_m)}}$$

and

$$\boxed{\mathbf{c = c_1{}^{2n}{}_1{}^{\pm p1} * c_2{}^{2n}{}_2{}^{\pm p2} * \ldots * c_n{}^{p2n}{}_n{}^{\pm pn} (* e_1{}^{2r}{}_1 * e_2{}^{2r}{}_2 * \ldots * e_s{}^{2r}{}_t)}}$$

where the terms in parentheses are optional for both *b* and *c*. With other words, in plain English that means if b contains any factor at any power, then c must contain the same factor at an even power ± the power of *b*'s factor. In addition, either *b* or *c* can contain any other factor as long as its power (exponent) is an even number. Furthermore, it means there are actually an infinity of sets/vectors (*a, b, c*) - we have two *a*s for any couple of *b* and *c* - of natural numbers of the above form that satisfies the given identity.

For example, let:

$$b = 2^3 * 5 = 8 * 5 = 40$$

thus:

$$c = 2*5^3 = 2*125 = 250$$

Thus:

$$bc = 2^4*5^4 = 10^4 = 10000$$

Therefore:

$$\sqrt{bc} = \sqrt{2^4*5^4} = 2^2*5^2 = 100$$

Thus:

$$a_{1,2} = b + c \pm 2\sqrt{bc} = 40 + 250 \pm 2*100 = 40 + 250 \pm 200 = 490 \text{ or } 90$$

Now:

a, b, and *c* must satisfy the original equality:

$$\sqrt{a} = \sqrt{b} + \sqrt{c}$$

So, substituting the above found values for $a = 490$, $b = 250$ and $c = 40$ we have:

$$\sqrt{490} = \sqrt{40} + \sqrt{250}$$

or,

$$7\sqrt{10} = 2\sqrt{10} + 5\sqrt{10}$$

which is true.

Now, let's substitute the above found values for $a = 90$, $b = 250$ and $c = 40$ we have:

$$\sqrt{90} = \sqrt{40} + \sqrt{250}$$

or,

$$3\sqrt{10} = 2\sqrt{10} + 5\sqrt{10}$$

Which does not verify!?... But, this is true:

$$3\sqrt{10} = 5\sqrt{10} - 2\sqrt{10}$$

Which means two things:

1) Only the $a_1 = b + c + 2\sqrt{bc}$ solution satisfy this identity:

$$\sqrt{a} = \sqrt{b} + \sqrt{c}$$

$$(**) \quad \boldsymbol{a_1 = b + c + 2\sqrt{bc}}$$

2) $a_2 = b + c - 2\sqrt{bc}$ solution satisfy this identity:

$$\sqrt{a} = \sqrt{b} - \sqrt{c}$$

$$(***) \quad \boldsymbol{a_2 = b + c \pm 2\sqrt{bc}}$$

2.3 FURTHER EXPLORATION

Now, let's go back to the initial identity:

$$\sqrt{a} = \sqrt{b} + \sqrt{c}$$

And replace + by -, so we have:

$$\sqrt{a} = \sqrt{b} - \sqrt{c}$$

So, is this new identity holding truth, too?

Taking the same approach as above, let be:

$$\alpha = \sqrt{a}$$

and

$$\beta = \sqrt{b} - \sqrt{c}$$

Thus:

$$(\beta - \sqrt{b})^2 = c$$

and:

$$\beta^2 - 2\sqrt{b}\,\beta + b = c$$

Now re-grouping the terms:

$$\beta^2 + (b - c) = 2\sqrt{b}\,\beta$$

Now, squaring both sides:

$$\beta^4 + 2\beta^2 (b - c) + (b - c)^2 = 4\ \beta^2 b$$

Regrouping the terms again:

$$\beta^4 - 2\beta^2 (b + c) + (b - c)^2 = 0$$

And the solutions for this quadratic equation are:

$$\beta^2 = ½\{ 2(b+c) \pm\sqrt{[4(b+c)^2 - 4 (b-c)^2]} \} \text{ or } \beta^2 = b+c \pm 2\sqrt{bc}$$

Thus, since:

$$\alpha = \sqrt{a}$$

and we must have:

$$\alpha = \beta$$

we have reach a similar conclusion for:

$$\sqrt{a} = \sqrt{b} - \sqrt{c}$$

like for:

$$\sqrt{a} = \sqrt{b} + \sqrt{c}$$

Thus, the similar relationship we must have between *a, b,* and *c*:

$$(***) \quad a = b + c - 2\sqrt{bc}$$

In conclusion, we can say now:

"Let *a, b,* and *c* be positive integers (*a = b = c = 0* also verify, as a trivial solution). The conditions on *a, b,* and *c* that satisfy the identity:

$$\boxed{\sqrt{a} = \sqrt{b} \pm \sqrt{c}}$$

are:

$$a_{1,2} = b + c \pm 2\sqrt{bc}$$
$$b = b_1^{p_1} * b_2^{p_2} * \ldots * b_n^{p_n} (* d_1^{2q_1} * d_2^{2q_2} * \ldots * d_m^{2q_m})$$
$$c = c_1^{2n_1 \pm p1} * c_2^{2n_2 \pm p2} * \ldots * c_n^{p2n_n \pm pn} (* e_1^{2r_1} * e_2^{2r_2} * \ldots * e_s^{2r_t})$$

where the terms in parentheses are optional for both *b* and *c*. In addition, for the '+' version of the identity, $a = b + c + 2\sqrt{bc}$ and for the '-' version of the identity, $b + c - 2\sqrt{bc}$ "

3.0 FERMAT's LAST THEOREM (with RADICALS)

Finally, the identity:

$$\sqrt{a} = \sqrt{b} \pm \sqrt{c}$$

inspired a rewording for the **Fermat's Last Theorem**[8], as follows:

"Is there *a, b,* and *c* natural numbers and *n* natural number greater than 2 ($n > 2$) – i.e., 3, 4, … - such that:

$$\mathbf{a^{1/n} = b^{1/n} + c^{1/n}}$$

or

$$\mathbf{a^{1/n} = b^{1/n} - c^{1/n}?}$$

or

$$\boxed{\mathbf{a^{1/n} = b^{1/n} \pm c^{1/n}?}}$$

From the above problems we know that for $n = 2$ we have an infinity of solutions (sets/vectors of *a, b*, and *c* satisfying that identity, same as there is an infinity of solutions for the **Pythagorean Theorem**[10] (that probably inspired **Fermat** to state his **Last Theorem**):

$$a^2 = b^2 + c^2$$

[By the way, there is an infinity of **Pythagorean** numbers that satisfy:

$a^2 = b^2 - c^2$ just re-arrange the values for *a, b* and *c*.]

3.1 ATTEMPT TO SOLVING FERMAT's LAT THEOREM (with RADICALS)

Now, for $n = 3$ the above identity becomes:

$$\mathbf{a^{1/3} = b^{1/3} + c^{1/3}}$$

and following the same method to eliminate the radicals (of degree 3 this time), I have arrived to the following equation where I can eventually express *a* as a function of *b* and *c*:

$$a^3 - 3(b+c)a^2 + (3b + 3c - 27bc)a - 54bc(b+c) = 0$$

and from here, using **Tartaglia (Cardano)**[9]'s formulas I can see what conditions *b* and *c* must meet in order for *a*, as a function of *b* and *c,* be an integer:

Now, given:

$$x^3 + \alpha x^2 + \beta x + \gamma = 0$$

Tartaglia (Cardano)'s formulas are:

$$x_{1,2,3} = \sqrt[3]{(q/^2 \pm \sqrt{(q^2/4 + p^3/27)})}$$

where:

$$p = \beta - \alpha^2/3$$

$$q = \gamma + (2\alpha^3 - 9\alpha\beta)/27$$

So, in our case I just need to replace:

α by $-3(b+c)$;

β by $(3b + 3c - 27bc)$;

and

γ by $-54bc(b+c)$.

But at this point I really need *Mathematica*[11] in order to arrive to formulas that express *a* as a function of *b* and *c*. And then to see if there is possible to find integers *b* and *c* such as *a* is also an integer.

Furthermore, assuming that exercise would be successful, it means that THIS version of Fermat's Last Theorem actually has solutions for $n > 2$, which at least to me would be AMAZING, proving, once again, that radicals are little wonders of the Arithmetic!

And, since in a dreaming mode, assume I can do the same for $n = 4$ (even though I imagine colossal computations to eliminate those radicals - even using *Mathematica* – we still have formulas to solve a degree 4 polynomial equation. But after that, even if elimination of radicals would be possible, there are not known formulas for solving a degree of 5 or higher of polynomial equation! **Thus, I guess "God was smart when after inventing the integers He stopped!"**

To be continued(?!...)

5.0 REFERENCES and BIBLIOGRAPHY

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